First draft

Multifractal Description of Lagrangian Field Theory

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Abstract

We show that the flow from the ultraviolet to the infrared sector of any high-dimensional nonlinear field theory approaches chaotic dynamics in a universal way. This result stems from for the dissipative effect of non-vanishing perturbations and implies that the infrared attractor of effective field theories replicates the geometry of *multifractal* sets. In particular, the Standard Model (SM) Lagrangian is characterized by a dominant generalized dimension $D_{SM} = 2$, while the same dimension of the Einstein-Hilbert Lagrangian turns out to be $D_{GR} = 4$. On the one hand, this finding disfavors any field-theoretic unification of SM and General Relativity (GR). On the other, it hints that the continuous spectrum of dimensions lying between D_{SM} and D_{GR} may naturally account for the existence of non-baryonic Dark Matter.

1. Introduction

Few theorists would dispute the compelling success enjoyed by the two pillars of contemporary science, the Standard Model of high-energy physics (SM) and General Relativity (GR). As *effective* field-theories, SM and GR describe remarkably well a wealth of phenomena, from sub-nuclear physics to the realm of astronomical scales and cosmology. However, several long-standing issues hint that either new physics *or* a deeper conceptual structure are required for a complete account of Nature beyond SM and GR [ ].

Recently, H. Nicolai has summarized the main foundational challenges confronting both the SM and GR [ ]. His critique targets the vastly uncharted territory lying beyond perturbative quantum field theory (QFT), as well as the inherent singularities of the strong gravity regime in GR:

"*But the real problem with the SM is theoretical: it is not clear whether it makes sense at all as a theory beyond perturbation theory, and these doubts extend to the whole framework of quantum field theory (QFT) (with perturbation theory as the main tool to extract quantitative predictions). The occurrence of "ultraviolet" (UV) divergences in Feynman diagrams, and the need for an elaborate mathematical procedure called renormalisation to remove these infinities and make testable predictions order-by-order in perturbation theory, strongly point to the necessity of some other and more complete theory of elementary particles.*

On the GR side, we are faced with a similar dilemma. Like the SM, GR works extremely well in its domain of applicability and has so far passed all experimental tests with flying colours, most recently and impressively with the direct detection of gravitational waves (see "General relativity at 100"). Nevertheless, the need for a theory beyond Einstein is plainly evident from the existence of space–time singularities such as those occurring inside black holes or at the moment of the Big Bang. Such singularities are an unavoidable consequence of Einstein's equations, and the failure of GR to provide an answer calls into question the very conceptual foundations of the theory."

In this work, we do not proceed along the path of Quantum Gravity, as suggested by Nicolai and many other researchers in the field. Working in the context of the *emergence* paradigm, we model the flow from the ultraviolet to the infrared regime of effective field theory starting from the universal behavior of far-from-equilibrium nonlinear dynamic systems. The underlying premise is that the asymptotic trajectory of any high-dimensional nonlinear dynamic system ends up on a *strange attractor*, under the steady influence of non-vanishing perturbations from equilibrium. The perturbations can be naturally associated with either primordial density fluctuations in the early Universe or unbalanced high-energy vacuum fluctuations of QFT. The bottom line of this approach is that *strange attractors form the infrastructure of Lagrangian field theory*. Further recalling that strange attractors, as fingerprints of chaos, bear resemblance to equilibrium statistical mechanics via ergodicity, global stability and invariant probability distributions [ ], we proceed to the study of effective field theory using the tools of *multifractal analysis*.

2. The long-term approach to chaos in nonlinear dynamics

The downward flow of a primary system of variables describing the ultraviolet sector may be mapped to a system of differential equations having the universal form

$$x'_{\tau} = f(x(\tau), \lambda(\tau), \tau, D(\tau)) \qquad (1)$$

Here, x is the vector of primary variables $x \equiv \{x_i\}$, $i = 1, 2, ..., n$ and λ, τ, D denote, respectively, the control parameters vector $\lambda = \{\lambda_j\}$, $j = 1, 2, ...m$, the evolution parameter and the dimension of the embedding space. If the dimension of the embedding space is taken to be an independent variable or a control parameter, the system (1) reduces to

$$x'_\tau = f(x(\tau), \lambda(\tau), \tau) \tag{2}$$

...

$$x(\tau) = x_s(\tau) + y(\tau) \tag{3}$$

...

$$y'_\tau = f(\{x_s + y\}, \lambda) - f(\{x_s\}, \lambda) \tag{4}$$

...

$$y'_\tau = \sum_j L_j(\lambda) y_j + h(\{y\}, \lambda) \tag{5}$$

...

$$z'_\tau = (\lambda - \lambda_c) - uz^2 \tag{6}$$

$$z'_\tau = (\lambda - \lambda_c)z - uz^3 \tag{7}$$

$$z'_\tau = (\lambda - \lambda_c)z - uz^2 \tag{8}$$

...

$$z'_\tau = [(\lambda - \lambda_c) + i\omega_0]z - uz|z|^2 \tag{9}$$

...

$$\lambda_l - \lambda_c \approx K\delta^{-l} \tag{10}$$

...

3. Multifractals: a concise overview

As it is known, the *box-counting dimension* defines the main scaling property of fractal structures and is a measure of their self-similarity. Multifractals are global mixtures of fractal structures, each characterized by its local box-counting dimension. Self-similarity of multifractals is accordingly defined in terms of a *multifractal spectrum* describing the overall distribution of dimensions. In the language of chaos and complexity theory, multifractal analysis is the study of *invariant sets* and is a powerful tool for the characterization of generic *dynamical systems*.

In the recursive construction of multifractal sets from $i = 1, 2..., N$ local scales r_i with probabilities p_i, the definition of the box-counting dimension leads to [ ]

$$\sum_{i=1}^{N} p_i^q r_i^{\tau(q)} = 1 \tag{11}$$

in which

$$\sum_{i=1}^{N} p_i = 1 \tag{12}$$

Here, q and $\tau(q)$ are two arbitrary exponents and the latter is typically presented as

$$\tau(q) = (1-q)D_q \tag{13}$$

where D_q plays the role of a *generalized dimension*.

The closure relationship (11) may be extended to a continuous distribution of scales in D - dimensional space time. It reads

$$\boxed{\int p^q(x)\, r^{\tau(q)}(x)\, d^D x = 1} \tag{14}$$

2. GR as topological analogue of SM

Consider now the field makeup of the SM, formed by 16 *independent* "flavors": two massive gauge bosons (W, Z), gluon (g), the Higgs scalar (H), neutrinos, charged leptons and quarks. The SM structure can be conveniently organized in the 4×4 matrix

$$SM = \begin{pmatrix} g & \nu_e & \nu_\mu & \nu_\tau \\ W & e & \mu & \tau \\ Z & u & c & b \\ H & d & s & t \end{pmatrix} \tag{15}$$

The photon (γ) is absent from (15) as it is built from the underlying components of the electroweak sector, whereby $\gamma = \gamma(W_\mu^3, B_\mu)$ and $B_\mu = B_\mu(W_\mu^3, Z)$ [ ].

It was shown in [ ] that, near the electroweak scale M_{EW}, the spectrum of particle masses m_i entering the SM satisfies the "closure" relation

$$\sum_{i=1}^{16} (\frac{m_i}{M_{EW}})^2 = 1 \tag{16}$$

It is apparent that (15) shares the same formal structure with the metric tensor of GR, that is,

$$GR = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} \tag{17}$$

where there are only 10 independent entries under the standard assumption $g_{\mu\nu} = g_{\nu\mu}$. Starting from the GR definitions of interval and proper time leads to ($c = 1$)

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu} \frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = 1 \tag{18}$$

subject to the constraint

$$\sum_{\nu=0}^{3} g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho} = \begin{cases} 1, & \mu = \rho \\ 0, & \mu \neq \rho \end{cases} \tag{19}$$

Comparing ( ), ( ) and ( ) reveals the following mapping

$$\boxed{GR : (\, p_i \Rightarrow g^{\mu\nu} g_{\nu\rho} ,\; q = 1/2 \;,\; D_q = 4 \,,\; \tau(q) = 2)}$$

$$\boxed{SM : (\, p_i \Rightarrow 1,\; q = 0,\; D_q = \tau(q) = 2 \,)} \tag{20}$$

It is instructive to note that $D_0 = 2$ coincides with the fractal dimension of quantum mechanical paths in free space [ ], whereas $D_{1/2} = 4$ recovers the four-dimensionality of geodesic paths in classical spacetime.

A couple of conclusions may be drawn from (20):

- GR may be viewed as topological analogue of the SM, defined by a half-unitary exponent q and a dimension that is twice the SM dimension (that is, $D_{1/2} = 2D_0$).

- The spectrum of particle *mass scales* (m_i / M_{EW}) and the *four-vector of differential coordinates* ($dx^\mu / d\tau$) form the basis for the multifractal description of SM and GR, respectively.

3. Multifractal formulation of effective field theories

Effective Lagrangians in QFT may be described as sums of polynomial terms having the generic form

$$L(\varphi,\partial\varphi) = \sum_{i,k,l,m,n} [c_{i,i}(\partial\varphi)^k + c_{i,i+1}(\partial\varphi)^l(\varphi)^m + c_{i+1,i+1}(\varphi)^n] \tag{21}$$

To simplify notation, we focus below on the basic unit entering the sum (21), namely on

$$L_u(\varphi,\partial\varphi) = c_{11}(\partial\varphi)^k + c_{12}(\partial\varphi)^l(\varphi)^m + c_{22}(\varphi)^n \Rightarrow c_{11}z_1^k + c_{12}z_1^l z_2^m + c_{22}z_2^n = 1 \tag{22}$$

in which

$$z_1^k = \frac{(\partial\varphi)^k}{L(\varphi,\partial\varphi)},\ z_1^l = \frac{(\partial\varphi)^l}{\sqrt{L(\varphi,\partial\varphi)}},\ z_2^m = \frac{\varphi^m}{\sqrt{L(\varphi,\partial\varphi)}},\ z_2^n = \frac{\varphi^n}{L(\varphi,\partial\varphi)} \tag{23}$$

$c_{1,2,3}$ are constants at given setting, for example, at a given energy scale. Therefore,

$$\boxed{r_{11}^k + r_{12}^l r_{21}^m + r_{22}^n = 1} \tag{24}$$

where

$$r_{11}^k = c_{11} z_1^k,\ r_{12}^l = \sqrt{c_{12}}\, z_1^l,\ r_{21}^m = \sqrt{c_{12}}\, z_2^m,\ r_{22}^n = c_{22} z_2^n$$

If $c_{1,2,3}$ depend on the field content or their derivatives, (24) assumes the general form

$$\boxed{c_{11}^{q_1}(r_{11})\, r_{11}^k + c_{12}^{q_2}(r_{12}, r_{21}) r_{12}^l r_{21}^m + c_{22}^{q_3}(r_{22}) r_{22}^n = 1} \tag{25}$$

where $q_{1,2,3}$ are non-vanishing exponents and

$$c_{11} + c_{12} + c_{22} = 1 \tag{26}$$

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4. GR as multifractal set

Einstein-Hilbert action:

$$S = \int R\sqrt{-g}\, d^4x \tag{27}$$

…

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu}(\Gamma^{\sigma}_{\mu\sigma,\nu} - \Gamma^{\sigma}_{\mu\nu,\sigma} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\sigma}_{\sigma\rho} + \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}) \tag{28}$$

…

$$g^{\mu\nu}(\Gamma^{\sigma}_{\mu\sigma,\nu} - \Gamma^{\sigma}_{\mu\nu,\sigma})\sqrt{-g} = 2L_G\sqrt{-g} \tag{29}$$

…

$$L_G = \frac{dS}{\sqrt{-g}d^4x} = g^{\mu\nu}(\Gamma^{\sigma}_{\mu\nu}\Gamma^{\sigma}_{\sigma\rho} - \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}) \rightarrow g^{\mu\nu}\,\frac{\Gamma^{\sigma}_{\mu\nu}\Gamma^{\sigma}_{\sigma\rho} - \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}}{L_G} = 1 \tag{30}$$

…

$$\sum_{\nu=0}^{3} g^{\mu\nu} g_{\nu\rho} = \delta_{\rho}^{\mu} = \begin{cases} 1, & \mu = \rho \\ 0, & \mu \neq \rho \end{cases} \quad (31)$$

…

5. SM as multifractal set

SM Lagrangian

$$L_{SM} = -\frac{1}{4}\sum_{V} V_{\mu\nu}^{a} V^{a\mu\nu} + \overline{f_L^i} i\gamma^{\mu} D_{\mu} f_L^i + \overline{f_R^i} i\gamma^{\mu} D_{\mu} f_R^i + (Y_{ij}\overline{f_L^i} H f_R^j + h.c.) + (D^{\mu}H)^{\dagger}(D_{\mu}H) - V(H) \quad (32)$$

Here, the summation convention over repeated indices is assumed, with $(i, j) = 1, 2, 3$ extending over the three fermion families [ ]. The vector fields V corresponds to the three gauge groups of the SM, namely $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$,

$$V = \left\{ B, W^{a=1,2,3}, G^{a=1....8} \right\} \quad (33)$$

to which we associate the field-strength tensors

$$V_{\mu\nu}^{a} = \partial_{\mu} V_{\nu}^{a} - \partial_{\nu} V_{\mu}^{a} + g f_{abc} V_{\mu}^{b} V_{\nu}^{c} \quad (34)$$

and covariant derivative operators

$$D_{\mu} = \partial_{\mu} - i\sum_{V} g_V t_V^a V_{\mu}^a \quad (35)$$

The last couple of terms denote the kinetic and potential contributions of the Higgs field,

$$V(H) = -m_H^2 H^\dagger H + \lambda (H^\dagger H)^2 \tag{36}$$

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Appendix

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Quantum Mechanics	Standard Model	General Relativity
Invariance under changes in representation	Invariance under local gauge transformations	Diffeomorphism invariance

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References

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